**Ornstein-Uhlenbeck Process**

So the Wiener process forms the backbone of the Ornstein-Uhlenbeck process, which seems to adequately describe all the transfer matrix modeling going on. Recall our vector Wiener process.



Now we’ll move on to:

**Fokker-Plank Equations**

Now let’s consider how to obtain an evolution equation for the probability distribution function P(X(t)). Formally all we have to do is take the average of F(X(t)), where F(X(t)) = δ(ξ – X(t)). But in its present form, the differential equation for F(X(t)) is inconvenient from the standpoint of evaluating expectations because some of the **X**’s are evaluated right in the middle of a d**W** interval, which makes the two correlated. As we said above, this can be useful if we’re trying to solve for X, or F(X), say. But if we want to be able to just solve for <F(X(t))>, then our differential equation has the just-mentioned drawbacks. What we want is for all **X** terms to be evaluated at the beginning of the interval, i.e., tm. So starting from the original equation,



let’s consider the first term in dF. This time we need to do the opposite of what we did last time, shift all the **X**’s in the d**X** term to tm. Doing so out to order O(dt),



Replacing the **W** product with its average, we get:



We may with impunity to O(dt) change the evaluation point of the **X**’s in the last term to tm, and then we may replace the **W** product with its average.



Altogether then, we have:



Now the d**W**’s are decoupled from all the **X**’s. Taking the average over all variables, which is implicitly,



and dividing by dt we get:



where,



We’ll note that the Ito version is similar to the previous section’s result. Even if the equation cannot be solved, a Taylor series expansion of the statistic can be calculated, via repeated application of the H-operator:



With this result in hand, we can write down the evolution equation for the probability distribution function: P(**ξ**,t) = <δ(**ξ** – **X**(t))>. Filling δ(**ξ** – **X**(t)) into F(**X**(t)). Like before, we will get:



and integrating by parts, we find:



where,



Let’s look at this from a different perspective, as is done in the Brownian motion file. Consider some F(X). How would its expectation evolve with time? Let’s look at it from the Ito perspective:



Now,



So filling this in…



I don’t think we could in general separate the average over the F and ΔX terms. For instance, we can’t say <X(t)X(tʹ)> = <X(t)><X(tʹ)> just because tʹ > t. But in an Ito process, the X’s in ΔX are evaluated at time t. So we can pull the X(s)’s out of the integral, as we basically have:



since we’ll recall from the Wiener process file that W(t+τ) – W(t) = W(τ). So we have:



Then doing the averages, we find:



Finally we can divide both sides by τ:



And this matches our more general result above for the Ito case λ = 0. Might also compare this FP equation to what we got in the general stochastic processes file.



We can see from the form of this equation, in retrospect, that we were presuming an Ito process λ = 0 back then as well.

**Equation for the particle density**

Say we have our equation for the probability distribution of the coordinates/particles (setting λ = 0 for the sake of discussion):



We can get moments of the distribution quite simply from this, or by using the prior ∂<F>/∂t = <HF> equation, for F = ξ, or ξ2, etc. But how can we get an equation for the density n(ξ)?

**Example**

The DMPK equation (1D) says:



What is an SDE for λ? Well, we need to put it in the form,



So let’s work out the derivative (choosing λ = 0 for Ito process),



Just going to set D = 1. And then, comparing, we see we need,



These imply,



So our SDE would read,



Using,



we can convert our Ito SDE to a Stratonovich SDE,



So,



In the large λ limit, this goes to:



It basically follows that we’d get a ln-normal distribution for λ in this limit (see OU SDE file). In the small λ limit, we’d have:



**Example**

Let’s do the same with the full DMPK,



(where t = 2L/ℓ(N+1)) And we compare to:



(Stratonovich term set to zero) I’ll set Dkℓ= 1.



Looks like the b’s index dependence requires k = ℓ, and j = i. So,



Comparing to,



So we need,



which comes to:



And is,



So our SDE would be:



If λi >> λi-1 >> λi-2 etc. >> 1, then we can say:



So again we see the log-normal distribution.

**Example**

Let’s do GDMPK,



Comparing to:



We have:



Working this out,



and,



So our equation is:



Suppose that the eigenvalues were bunched together, with average separation Δλ. Then, it would look like,



How does this behave? Can’t use perturbation theory per se´ because the zeroth order term would make λi = λk, and so the perturbation would be singular. So maybe use ‘singular’ perturbation theory? Well, at least this shows again how the perturbative expansion done by GF’s require that non-analytic approach.